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LETTER TO THE EDITOR

A note on classical motions under strong constraints

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Abstract. Strong, non-homogeneous constraints in full configuration space induce an extra potential term in restricted configuration space. This well known fact is interpreted in the light of recent results of Jackiw and of Gozzi and Thacker about Berry's phases and Hannay's angles for coupled systems. The indeterminacy caused by resonances, first pointed out by Takens, is partially lifted using an estimate by Grimshaw and Allen. We discuss implications for time-reversal symmetry breakdown. Slowly time-varying strong potentials, and quantum mechanical counterparts are also discussed briefly.

Let M^n be a manifold with a Riemannian metric and denote by T its kinetic energy. A *strongly constraining* potential is a function on total configuration space M of the form $\lambda W(x)$, $x \in M$, $\lambda \gg 1$, such that W vanishes on a submanifold $S^k \subset M$ and $W > 0$ on $M - S$. Rubin and Ungar [1] and Takens [2] studied natural mechanical systems $L = T - V - \lambda W$ on M. At the limit $\lambda \to \infty$ one does not get just the reduced Lagrangian $L_S = T_S - V_{|S}$ on TS, where the kinetic energy is that of the induced metric of the embedding of S into M: if W is non-uniform, there appears an extra potential, a well known fact in the motion of charged particles in magnetic fields [3].

The first purpose of this letter is to show how the results in [1, 2] can be obtained in a quick way, using results by Jackiw [4] and Gozzi and Thacker [5] about Berry's phase [6] and Hannay's angles [7] for coupled systems. Moreover, we will give a simple interpretation for the extra potential term.

For completeness, we first recall an averaging procedure for slow-fast classical systems, following the presentation in [8]. In quantum mechanics these ideas go back to the Born-Oppenheimer method [4]. The slow dynamics can be approximately decoupled from the fast. Besides the averaged potential, the averaged fast motion also introduces a 'magnetic-like' or 'gauge' effect in the slow phase space, through the appearance of 'anomalous commutators'; the classical counterpart was first described in [5] by a semiclassical 'dequantisation' procedure.

Let $H = p^2/2m + P^2/2 + V(q, Q)$ where $m \ll 1$, so that (p, q) is the fast motion. Assume that for each frozen Q the partial Hamiltonian $h = p^2/2m + V(q, Q)$ is completely integrable, so there are (partial) action-angle variables (I, θ) and canonical transformations

$$p = p(I, \theta; Q, m) \qquad q = q(I, \theta; Q, m) \tag{1}$$

for which the partial Hamiltonian is $h = K(I; Q, m)$. However, the mapping $(I, \theta, P, Q) \to (p, q, P, Q)$ is no longer canonical: the symplectic form [9]

$$\mathrm{d}P \wedge \mathrm{d}Q + \mathrm{d}p \wedge \mathrm{d}q$$

pulls back as

$$w = \mathrm{d}P \wedge \mathrm{d}Q + \mathrm{d}I \wedge \mathrm{d}\theta + (I, Q)\, \mathrm{d}I \wedge \mathrm{d}Q + (Q, \theta)\, \mathrm{d}Q \wedge \mathrm{d}\theta + \sum (Q_i, Q_j)\, \mathrm{d}Q_i \wedge \mathrm{d}Q_j \tag{2}$$

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where $(u, v) = p_u q_v - p_v q_u$ denote Lagrange brackets. It follows that the exact equations of motion for the full Hamiltonian $H = p^2/2 + K(I, Q; m)$ with respect to the pulled-back form are

$$\begin{aligned} \dot{I} &= (\theta, Q)P & \dot{\theta} &= K_I + (Q, I)P \\ \dot{Q} &= P & \dot{P} &= -K_Q + (Q, Q)P + (Q, \theta)K_I. \end{aligned} \tag{3}$$

The terms (Q, θ) in the symplectic form drop out under averaging with respect to the angle variables. Thus the *averaged* equations for the slow motion are

$$\dot{Q} = P \qquad P = -K_Q + \langle Q, Q \rangle P \tag{4}$$

where $\langle Q, Q\rangle$ is a 'magnetic-like force', since it exerts no work. The reduced Hamiltonian for the slow motion is $H = p^2/2 + K(Q; I, m)$ with I, the actions of the fast motions, acting as a parameter. The averaged, reduced symplectic form is

$$\langle w \rangle = \mathrm{d}P \wedge \mathrm{d}Q + \sum \langle Q_i, Q_j \rangle \, \mathrm{d}Q_i \wedge \mathrm{d}Q_j. \tag{5}$$

In this approximation, given a solution curve $(P(t), Q(t))$ of the decoupled slow subsystem, the motion of the fast variables can be recovered from the partial canonical transformation making I = constant and

$$\theta(t) = \theta_0 + \int^t K_I(I, Q(t)) \, \mathrm{d}t + \int^t \langle Q, I \rangle P(t) \, \mathrm{d}t. \tag{6}$$

The second term is a set of 'dynamic' phases, while the third, a lower-order correction, is a set of 'geometric phases', that is, Hannay's angles for the slow–fast system.

Consider now a strongly constrained system $H = T + (V + \lambda W)$. Choose a local chart for M with 'adapted coordinates' $y = (Q, q)$, where $Q \in R^k$ parametrises S and $q \in R^p (k + p = n)$ parametrises a tubular neighbourhood of S in M. Let (P, p) be the momenta obtained under Legendre transformation of $y = (Q, q)$ via T, so that the kinetic energy becomes

$$T = \tfrac{1}{2}{}^t PA(y)P + \tfrac{1}{2}{}^t pB(y)p + {}^t pC(y)P. \tag{7}$$

We may assume that ${}^t pC(Q, 0)P = 0$, which means that the q-fibres leave S orthogonally. Let us freeze the slow coordinate Q. Collecting the more important contributions for the motion of the q subsystems, we get a quadratic partial Hamiltonian

$$h = \tfrac{1}{2}{}^t pB(Q)p + \tfrac{1}{2}\lambda \, {}^t q W^{(2)}(Q) q \tag{8}$$

where $W^{(2)}$ is the Hessian of W relative to q at $(Q, 0)$. We even observe that, by the Morse lemma with parameters, we may assume $W = W^{(2)}$. For each fixed Q this is a *linear oscillator*, so by elementary means one finds a (partial) canonical transformation $p = p(I, \theta; Q, \lambda)$, $q = q(I, \theta; Q, \lambda)$ such that

$$h = K(I, Q, \lambda) = \sum \sqrt{\lambda} \, w_i I_i \tag{9}$$

where $w_i = w_i(Q)$ are the frequencies of the linear system $(B, W^{(2)})$, assumed for the moment to be distinct. Denoting $J_i = \sqrt{\lambda} \, I_i$, by our previous discussion we obtain the limiting system on S given by

$$H_S = \tfrac{1}{2} T_S + (V_{|S} + \sum J_i w_i). \tag{10}$$

Thus one gets a simple interpretation for the extra potential: the constants J_i are the adiabatic invariants of this slow–fast system, and the w_i are the frequencies of the transverse oscillations, suitably normalised.

Although this derivation may be lacking in mathematical rigour compared with the detailed works [1, 2], we believe it gives some good physical insights. For instance, Benettin *et al* [10], using Nekhoroshev's perturbation theory, showed that the energy for transverse motion separates from the energy of the constrained system, for *exponentially large* times, at least in the homogeneous case. This result is consistent with the current understanding of the time validity of adiabatic invariants.

We also observe that since the terms (Q, Q) average out, there are no magnetic terms in the symplectic form (or equivalently, by a 'minimal-coupling' transformation, in the Hamiltonian, taking the standard symplectic form in the slow phase space). This is true because, when dealing with a family of linear *anisotropic* oscillators, the partial canonical transformations $p = p(I, \theta; Q, \lambda)$, $q = q(I, \theta; Q, \lambda)$ are given in terms of cosines and sines, respectively. In the Lagrange brackets, only products of sines with cosines appear. This implies $\langle Q, Q\rangle = 0$. For the same reason, $\langle Q, I\rangle = 0$ so here the geometric term in (6) vanishes. There are no classical adiabatic angles for the transverse motion; there can, however, still be interesting topological effects, about which we will comment at the end of this letter.

We can add yet another twist to the problem by considering strong, slowly time-dependent, constraining potentials $\lambda W(x, \varepsilon t)$, where now there are two parameters $\lambda \gg 1$, $\varepsilon \ll 1$. Here $W(\cdot, \tau)$ vanishes on a slowly time-varying submanifold $S(\tau) \subset M$ and $W(\cdot, \tau) > 0$ on $M - S(\tau)$, with $\tau = \varepsilon t$. Assume that all $S(\tau)$ are modelled on the same 'abstract' manifold $\mathscr{S}$. Consider a parametrisation $x = x(Q, q, \tau) \in M$ in terms of local 'adapted coordinates' $y = (Q, q)$ in $R^k \times R^p$ $(k + p = n)$ such that $x(Q, 0, \tau)$ parametrises $S(\tau)$. As before, we may assume, without loss of generality, that the fibres $x_q \dot{q}$ are orthogonal to $S(\tau)$ at $q = 0$. In these y coordinates, the kinetic energy of the time-dependent Lagrangian $L(x, \dot{x}, \varepsilon t) = T - V(x, \varepsilon t) - \lambda W(x, \varepsilon t)$ is

$$T(y, \dot{y}, \varepsilon t) = \tfrac{1}{2}{}^t\dot{y} G \dot{y} + \varepsilon\, {}^t f_1 \dot{y} + \tfrac{1}{2}\varepsilon^2\, {}^t f_2 \tag{11}$$

where $G(y, \tau) = (x_y, x_y)$ is $n \times n$ symmetric positive definite, $f_1(y, \tau) = (x_y, x_\tau)$ is $n \times 1$ and $f_2(y, \tau) = (x_\tau, x_\tau)$ is a scalar. The Legendre transformation

$$p_y = G\dot{y} + \varepsilon f_1 \tag{12}$$

yields the Hamiltonian

$$H = \tfrac{1}{2} p_y G^{-1} p_y + V(y, \tau) + \lambda W(y, \tau) - \varepsilon p_y G^{-1} f_1 + \mathrm{O}(\varepsilon^2). \tag{13}$$

For fixed λ and $\varepsilon \ll 1$ one can consider $V + \lambda W$ as a single 'moderate' potential which we denote again by V. In many examples, this time-dependent $V(x, \tau)$ becomes time independent when written in the y coordinates, that is $V(x, \tau) = V(y)$. This happens, for instance, when there is a Lie group of isometries acting on M (e.g. slowly rotating systems) for which the potential is *equivariant*. In this case, it is easy to see that the $\mathrm{O}(\varepsilon)$ perturbing term $p_y G^{-1} g$ (which is the source of Berry's QM-phase or Hannay's classical adiabatic angles) is equal to the momentum $J(p_y)$ (an element of the dual Lie algebra) applied to the infinitesimal generator of the motion [8]. Another interpretation, in terms of the notion of 'Cartan connections', is given in Marsden *et al* [11]. These approaches are intended to unify the treatment of many known examples, such as the ball in the hoop, the Foucault pendulum, isotropic oscillators on space curves, and the rotating elliptical billiard [8, 12].

If one makes $\lambda = \infty$, assuming that $W(y)$ is time independent, and satisfies the homogeneity conditions, then the Hamiltonian (13) reduces to

$$H_{\text{red}} = \tfrac{1}{2} p_Q G_{\text{ind}}^{-1} p_Q + V(Q) + H^1 \tag{14}$$

where $H^1 = -\varepsilon p_Q G_{\rm ind}^{-1} f_{\rm proj}$, $f_{\rm proj} = (x_Q, x_T)_M$ is an $O(\varepsilon)$ term. For instance, when one deals with a submanifold S moving via rigid motions inside M, the last term becomes

$$H^1(p_s, X) = -p_s\,\mathrm{proj}_s(R_s)*X \qquad X = g^{-1}\,\mathrm{d}g/\mathrm{d}t \quad g = g(\varepsilon t) \quad X = 0(\varepsilon) \tag{15}$$

where R_s is the mapping $g \mapsto gs$ from the Lie group into M. This is the setting of [8] and [11].

We call attention to the simultaneous effect of the two quantities λ and ε when W is non-homogeneous. Physically, the system has three timescales: a 'fast' for the transverse motion, a 'normal' for the Q-motion, and a 'slow' due to the slow variation with respect to εt. Averaging with respect to the fast motion, then the reduced Hamiltonian (14) acquires an extra term, and is now both *space* and *time* dependent,

$$H_{\rm red} = \tfrac{1}{2} p_Q G_{\rm ind}^{-1}(Q, \varepsilon t) p_Q + V(Q, \varepsilon t) + \sum w_i(Q, \varepsilon t) J_i + H^1. \tag{16}$$

It was stressed by Takens that the averaging procedure breaks down at points $s \in S$ where there are *resonances* in transversal motion. At those points the extra potential becomes indefinite! This is, in fashionable language, a source of 'chaos'. We now present some indications, based on the relatively unknown but beautiful work by Grimshaw and Allen [13], that perhaps this indeterminacy could be (at least partially) lifted.

These authors deal with general quadratic Hamiltonians, with slowly time-independent coefficients, such that each frozen system has purely imaginary eigenvalues. For our purposes, it is sufficient to consider systems of the form (8), where here the matrices $B = B(Q)$ and $W = W(Q)$ are time dependent, through the *yet unknown* motion $Q = Q(t)$. Actually, one can pay attention just to the two degrees of freedom corresponding to the normal modes, say $i = 1, 2$, whose frequencies w_i come *near* to a collision. We recall that for symmetric matrices, eigenvalue collision is a codimension-*two* phenomenon, and here there is (for the time being) just one parameter, the time.

The sum of the actions $J_1 + J_2$ is still an adiabatic invariant, but a sizeable amount of action can be transferred between these resonant modes. The situation is well illustrated by the following model problem [14]:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}[(1+\varepsilon t)q_1^2 + (1-\varepsilon t)q_2^2 - 4\mu q_1 q_2] \tag{17}$$

for $|\varepsilon t| < \frac{1}{2}$. The frequencies $w_1 \geqslant w_2$ and angle ϕ from the smaller frequency mode to the q_1 axis are given by

$$w_{1,2}^2 = 1 \pm [(\varepsilon t)^2 + 4\mu^2]^{1/2} \qquad \tan\phi = (1/2\mu)[\varepsilon t + ((\varepsilon t)^2 + 4\mu^2)^{1/2}] \tag{18}$$

It follows that the eigendirections suffer a rotation of almost $\pi/2$, after the resonance process. This same feature carries over to the general case, for which the frequency behaviour is depicted in figure 1.

Grimshaw and Allen defined a coupling parameter $0 \leqslant k \leqslant \infty$, $r = \exp(-\pi k/2)$, by

$$k = (2\mu\rho)^2/\varepsilon|S_1 - S_2| \tag{19}$$

(in the example $\rho = 1 \quad S_1 = -S_2 = \frac{1}{2}$).

The exchange of action between the modes is estimated by [13]

$$\begin{aligned} &J_1^{\rm f} = (1-r)J_1^{\rm in} + rJ_2^{\rm in} - L \qquad J_2^{\rm f} = rJ_1^{\rm in} + (1-r)J_2^{\rm in} + L \\ &L^2 \leqslant 4(J_1^{\rm in} J_2^{\rm in}) r(1-r). \end{aligned} \tag{20}$$

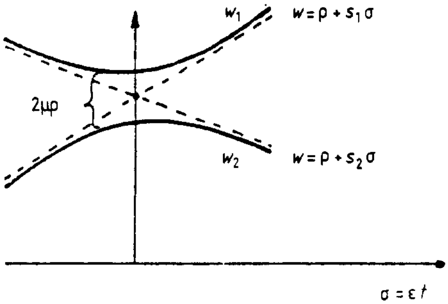

Figure 1. Generic near resonance in linear time-dependent Hamiltonians.

The indeterminacy term L vanishes if one of the initial actions is zero, or if $k=0$, $k=\infty$. If μ is fixed and $\varepsilon \to 0$, then $k=\infty$, $r=0$, so that $J_1^{\mathrm{f}} = J_1^{\mathrm{in}}$ and $J_2^{\mathrm{f}} = J_2^{\mathrm{in}}$. In practice, if $k > 1.72$, less than $1/100$ of actions is transferred. However, the eigendirections interchange, so one observes a striking physical behaviour. If on the other hand, ε is fixed and $\mu \to 0$, then $k=0$, $r=1$, so $J_1^{\mathrm{f}} = J_2^{\mathrm{in}}$ and $J_2^{\mathrm{f}} = J_1^{\mathrm{in}}$. The actions completely interchange, but this is compensated by the geometric interchange of eigenmodes.

The results of Grimshaw and Allen's can partially lift Takens chaos. Suppose that a trajectory $Q = Q(t)$ in restricted configuration space passes closest to a resonance at $t=0$. This distance is proportional to the parameter μ in figure 1, where $\mu = 0$ corresponds to exact resonance. The square root $\sqrt{\lambda}$ of the strength of the transverse potential is inversely proportional to κ, while the speed $|\dot{Q}|$ at $t=0$ is directly proportional to it.

This trajectory $Q(t)$ solves a time-dependent Hamiltonian given by (10), $H_S = \frac{1}{2}T_S(Q, \dot{Q}) + (V_{|S}(Q) + \Sigma\, J_i(t) w_i(Q))$, where all but two of the J_i (say J_1, J_2) are constant. On the other hand, the variations of J_1, J_2 are determined by a linear time-dependent system whose coefficients depend on the unknown Q. So, these equations are intrinsically coupled.

Nevertheless, in a first attack on the problem, it is reasonable to assume that the exchange between J_1 and J_2 takes place *instantaneously* at $Q(0)$, as $\mu \to 0$ and $\lambda \to \infty$, keeping $\dot{Q}(0)$ and the coupling parameter r fixed. The rationale is as follows: the natural timescale, from the point of view of the transverse oscillators, is the 'fast time' $\sigma = \sqrt{\lambda}\, t$. From [13, section 3] it follows that the action exchange takes place in a time interval of order $1/\lambda^{1/4}$.

Fixing $(J_1^{\mathrm{in}}, J_2^{\mathrm{in}})$, and taking into account that the indeterminacy is due to the term L in (20), one finds that through $(Q(0), \dot{Q}(0))$ there emanates a one-parameter *pencil* of trajectories! The *boundaries* of this pencil are found by solving (10) for $t \geqslant 0$ with values of J equal to J_1^{f}, J_2^{f} according to (20), taking $L = \pm 2[J_1^{\mathrm{in}} J_2^{\mathrm{in}} r(1-r)]^{1/2}$.

What is the origin of this non-uniqueness? Going back to Grimshaw and Allen's paper, one sees that the exact value of L depends on an angle-variable initial condition, which is usually disregarded in the averaging method. The phase difference between the modes w_1 and w_2, irrelevant for (10), now has great influence. Its removal must be paid for in terms of non-deterministic behaviour. In a certain sense, this provides a mechanism for *breaking down the time-reversal symmetry* within the framework of classical mechanics.

Some final remarks.

(i) If Q has one degree of freedom, the *exact* resonances are non-generic. But even so, if we allow time dependence as in (16), collisions of two frequencies are

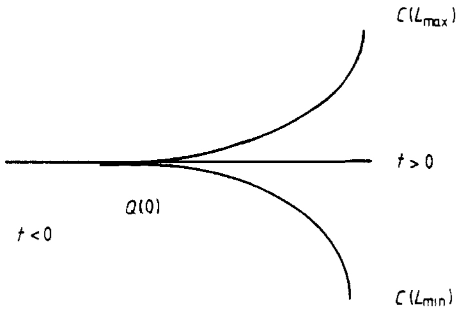

Figure 2. A mechanism for non-deterministic behaviour of solutions and time-reversal symmetry breaking within classical mechanics.

generically unavoidable. If Q has high enough degrees of freedom, then collisions (or near collisions) of *three* or more frequencies become unavoidable. So far, this latter situation has not yet been studied [2].

(ii) If there are no exact resonances, then the normal bundle of the embedding $S \subset M$ is trivialisable. So the topological invariants of restricted configuration space S can force the existence of frequency collisions.

(iii) The topological effect we alluded to earlier is a classical counterpart of Berry's diabolicity. Assume that a trajectory in configuration space makes a loop around an 'umbilic point' (a point such that two frequencies are equal). The loop does not need to be a periodic trajectory; that is, we do not impose that initial and final velocities coincide. If the loop is close enough to the umbilic, action exchanges do take place [15], but here we are interested in the adiabatic limit, so the actions remain invariant. Suppose we were able to compare two transverse oscillators, one with the parameters frozen at the initial value, the other corresponding to the slowly varying parameters. Then, at the end of the loop, each of the two modes involved will be in *opposition* of phase.

(iv) In statistical mechanics, time-reversal symmetry of Boltzmann equations is broken by a suitable averaging process. In a sense, the mechanism depicted in figure 2 is a classical mechanics analogue.

(v) The subtle effects of degeneracy *vis à vis* adiabaticity in quantum mechanics have been recently tackled by Gingold [16]. We think that it would be interesting to work out the quantum counterparts of [15, 16] (see also [17, 18]).

I wish to thank Professor Alan Weinstein for having called attention to Takens' paper in this office, back in 1982. I even recall him drawing something like figure 2 on the blackboard. This work was supported in part by CNPq/Brazil grant 30.0007-83.

References

[1] Rubin H and Ungar P 1957 Motion under a strong constraining force *Commun. Pure Appl. Math* 65–87
[2] Takens F 1979 *Motion under the influence of a strong constraining force* (*Lecture Notes in Mathematics* **819** (Berlin: Springer)
[3] Spitzer Jr L 1956 *Physics of Fully Ionized Gases* (New York: Interscience)
[4] Jackiw R 1987 Three elaborations on Berry's connection, curvature and phase *preprint* CTP-MIT 1529

[5] Gozzi E and Thacker W D 1987 Classical adiabatic holonomy and its canonical structure *Phys. Rev.* D **35** 2398-406
[6] Berry M V 1984 Quantal phase factors accompanying adiabatic changes *Proc. R. Soc.* A **392** 45-57
[7] Hannay J H 1985 Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian *J. Phys. A: Math. Gen.* **18** 221-30
[8] Koiller J 1989 Classical adiabatic angles for slowly moving mechanical systems *AMS Contemporary Math.* **97** 159-85
[9] Abraham R and Marsden J E 1978 *Foundation of Mechanics* (Reading, MA: Benjamin-Cummings)
Arnol'd V I 1978 *Mathematical Methods of Classical Mechanic* (Berlin: Springer)
[10] Bennetin G, Galgani L and Giorgilli A 1987 Realisation of holonomic constraints and freezing of high degrees of freedom in the light of classical perturbation theory: Part 1 *Commun. Math. Phys.* **113** 87-104; 1989 Part 2 *Commun. Math. Phys.* **121** 557-602
[11] Marsden J E, Montgomery R and Ratiu T 1990 Reduction, Symmetry and Berry's phase in Mechanics *AMS Memoirs* to appear
[12] Kuglen M and Strikman S 1988 Berry's phases, locally inertial frames, and classical analogues *Phys. Rev.* D **37** 934-7
[13] Grimshaw R and Allen J S 1979 Linealy coupled, slowly varying oscillators *Stud. Appl. Math.* **61** 55-71
[14] Pinto A Sá, Vieira G R and Koiller J Numerical experiments in linear mode conversion *preprint* LNCC/CNP_q
[15] Koiller J Hannay's angles outside the adiabatic range: coupled linear oscillators near resonance *J. Math. Phys.* to appear
[16] Gingold H 1987 In general, the less degeneracy the less transition. A principle for time-dependent Hamiltonian systems in quantum mechanics *J. Math. Phys.* **28** 2400-6
[17] da Costa R C T 1982 Constraints in quantum mechanics *Phys. Rev.* A **25** 2893-900
[18] da Costa R C T 1983 Constrained particles in quantum mechanics *Lett. Nuovo Cimento* **36** 393-6